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ON BRANCHED COVERINGS OF SOME HOMOGENEOUS SPACES

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Dedicated to the memory of Prof. Michael Schneider

We study nonsingular branched coverings of a homogeneous space X. There is a vector bundle associated with such a covering which was conjectured by O. Debarre to be ample when the Picard number of X is one. We prove this conjecture, which implies Barth–Lefschetz type theorems, for lagrangian grassmannians, and for quadrics up to dimension six. We propose a conjectural extension to homogeneous spaces of Picard number larger than one and prove a weaker version.

1. INTRODUCTION AND MAIN RESULTS

Let $f: Y \to \mathbb{P}^n$ be a branched covering of degree d of a complex projective space where Y is a nonsingular connected complex projective variety. A celebrated result of R. Lazarsfeld [11] states the following:

the induced morphism $f_*: H^i(\mathbb{P}^n, \mathbb{C}) \to H^i(Y, \mathbb{C})$ *is an isomorphism for* $i \leqslant n - d + 1$.

There is a natural vector bundle $\mathscr{E}$ on $\mathbb{P}^n$ of rank $d-1$ associated with f, defined by the splitting

$$f_* \mathscr{O}_Y \cong \mathscr{O}_{\mathbb{P}^n} \oplus \mathscr{E}^*$$

induced by the trace homomorphism. The map f factors through an embedding of Y in the total space $|\mathscr{E}|$ of $\mathscr{E}$. Lazarsfeld proved that $\mathscr{E}$ is always *ample*.

This result implies in particular W. Barth's theorem [2] for the cohomology of small codimensional subvarieties of projective spaces. Since Barth's fundamental paper, much attention has been paid to small codimension subvarieties of more general complex projective homogeneous spaces [20, 5, 1]. This and the result of Lazarsfeld lead to the question of understanding low degree branched coverings of such spaces $X = G/P$, where G is a semisimple complex Lie group and P is a parabolic subgroup. Note that the codimension $\operatorname{cod}(X, |\mathscr{E}|) = d - 1$. A recent paper of Debarre [4] gives strong evidence to the following conjecture:

CONJECTURE. *Let $f: Y \to X = G/P$ be a branched covering where Y is a nonsingular connected complex projective variety, and $Pic(X) = \mathbb{Z}$. Then the associated vector bundle $\mathscr{E}$ on X is ample.*

The assumption $Pic(X) = \mathbb{Z}$ simply means that P is a maximal parabolic subgroup. For $G = SL(n, \mathbb{C})$, the homogeneous space G/P is then a grassmannian. The first half of the way to the conjecture in that case was made in [9], the second half in [15]. The purpose of this paper is to prove the conjecture for a few other cases. Our main result is the following:

THEOREM A. *Let $\mathbb{LG}_n$ be the lagrangian grassmannian of maximal isotropic subspaces of a 2n-dimensional symplectic vector space. Then if $f\colon Y \to \mathbb{LG}_n$ is a branched covering where Y is a nonsingular connected complex projective variety, then the associated vector bundle $\mathscr{E}$ on $\mathbb{LG}_n$ is ample.*

Here, $\mathbb{LG}_n$ is homogeneous with $G = Sp(2n, \mathbb{C})$, the symplectic group; with the exception of $\mathbb{P}^{2n-1}$, $\mathbb{LG}_n$ is the only homogeneous space for the symplectic group which is a hermitian symmetric space. Using Lazarsfeld's ideas, we immediately deduce from Theorem A the following Barth–Lefschetz-type theorem:

THEOREM B. *Let $f\colon Y \to \mathbb{LG}_n$ be a branched covering of degree d where Y is a nonsingular connected complex projective variety. Then the induced morphism*

$$f^*: H^i(\mathbb{LG}_n, \mathbb{C}) \to H^i(Y, \mathbb{C})$$

is an isomorphism for $i \leqslant \dim \mathbb{LG}_n - d + 1$.

The cohomological range of A. Sommese's Barth–Lefschetz-type theorem [19, 20] concerning a submanifold of a homogeneous space $X = G/P$ depends on the k-ampleness (this is a suitable weakening of the notion of ampleness) of the normal bundle of the submanifold. To boot, by considering N. Goldstein's computation [7] of the k-ampleness of the tangent bundle of X, we can see that the codimension above which one loses all cohomological information depends linearly on the rank of G, that is on the dimension of a maximal torus contained in G; this is reminiscent of G. Faltings' connectedness result for homogeneous spaces (see [5, 7] for the details). Here, our condition involves the dimension of $X = G/P$, which is in general much larger, although in the case of $\mathbb{P}^n$ this makes no difference.

After the one of projective spaces, the easiest case should be the case of quadrics. Surprisingly enough, it does not seem to be so, despite partial results of the first author [10]. Actually, we have been able to prove the conjecture only for quadrics of very small dimension:

THEOREM C. *Let $\mathbb{Q}_n$ be the n-dimensional nonsingular quadric with $3 \leqslant n \leqslant 6$. If $f\colon Y \to \mathbb{Q}_n$ is a branched covering where Y is a nonsingular connected complex projective variety, then the associated vector bundle $\mathscr{E}$ on $\mathbb{Q}_n$ is ample.*

Finally, let us mention that the conjecture by Debarre should be extended to the cases of homogeneous spaces $X = G/P$ with P not necessarily maximal. In this case, one cannot expect the vector bundle associated with a branched covering to be ample. Nevertheless, it will be k-ample for a suitable value k; this implies Barth–Lefschetz-type theorems. In Section 5.3 we propose a related conjecture and give some evidence to it by proving the following result:

THEOREM D. *Let $f\colon Y \to X = G/P$ be a branched covering where Y is a nonsingular connected complex projective variety, and $G = SL(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$. Then the associated vector bundle $\mathscr{E}$ on X is generated by its global sections.*

Actually, as in [11] we prove a stronger positivity property for $\mathscr{E}$ on $X = G/P$ with Picard number one: $\mathscr{E} \otimes \mathcal{O}_X(-1)$ is generated by its global sections where $\mathcal{O}_X(1)$ is the (very) ample generator of $Pic(X)$. To show this we set up an appropriate Castelnuovo–Mumford-type criterion on X and use vanishing theorems.

In the case of lagrangian grassmannians, we have been obliged to get out of the algebraic category, and instead we use vanishing theorems “à la Nakano” for hermitian vector bundles. In Section 2, we will recall a few elementary facts on the curvature of hermitian bundles, and state a vanishing theorem that will be suitable to our purposes. With the help of a formula due to D. Snow for the curvature of homogeneous bundles, we will show in Section 3 how to apply this theorem to the proof of Theorem A. In Section 4 we deal with low-dimensional quadrics, where we make use of the so-called spinor bundles to obtain Theorem C. Section 5 is devoted to the proof of Theorem D, which is simple on ordinary flags, and much more involved in the symplectic case.

2. ON THE CURVATURE OF HERMITIAN VECTOR BUNDLES

2.1. Griffiths and Nakano positivity

Let E be a hermitian vector bundle on an n-dimensional nonsingular complex variety X. The curvature Θ_E of the associated Chern connection is a (1, 1)-form with values in the vector space of hermitian endomorphisms of E. For every point $z \in X$ and every local coordinate system $(z_j)_{1 \leqslant j \leqslant n}$ at z, the curvature can be written as

$$\Theta_E = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu,$$

with $c_{jk\lambda\mu} = \bar{c}_{kj\mu\lambda}$, where $(e_1, \ldots, e_r)$ is a local frame of E in a neighborhood of z. One may consider Θ_E as a hermitian form on $TX \otimes E$:

$$\Theta_E(u) = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}(z) u_{j\lambda} \bar{u}_{k\mu}$$

for $u = \sum_{j,\lambda} u_{j\lambda}(\partial/\partial z_j) \otimes e_\lambda \in T_z X \otimes E_z$.

Definition. The hermitian vector bundle E is Nakano (semi-) positive if Θ_E defines a (semi-) positive definite hermitian form on $TX \otimes E$. It is Griffiths (semi-) positive if Θ_E is (semi-) positive on non-zero decomposable tensors in $TX \otimes E$.

Nakano vanishing theorem states that the adjoint to a Nakano positive vector bundle on a compact Kähler manifold has no higher cohomology:

THEOREM 2.1 (cf. Nakano [16]). *Let E be a hermitian vector bundle on a compact Kähler manifold. Suppose that E is Nakano semi-positive, and that Θ_E is positive definite at least at one point. Then*

$$H^q(X, K_X \otimes E) = 0 \quad \forall q > 0.$$

2.2. A variant of Griffiths' vanishing theorem

We will need in the sequel of this paper a consequence of the Nakano vanishing theorem, which can also be seen as a variant of Griffiths' vanishing theorem [8]. We will state it in terms of *Schur powers*.

Let V be a finite-dimensional complex vector space. To each partition λ, i.e. to every non-increasing sequence of non-negative integers $\lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant 0$ (= the *parts* of λ), one associates a Schur power $S_\lambda V$, which is a polynomial $Gl(V)$-module [14]. This module reduces to zero if and only if the number of nonzero parts of λ (= the *length* $l(\lambda)$) is larger than the dimension of V. Otherwise, $S_\lambda V$ is an irreducible $Gl(V)$-module.

The usual symmetric powers are a special case of Schur powers. They correspond to partitions with a single nonzero part, or length one partitions. Also the wedge powers correspond to partitions with all nonzero parts equal to one. The tensor product of any Schur power with a symmetric or a wedge power is described by the classical Pieri's rules [14], which imply, in particular, that each Schur power $S_\lambda V$ is an irreducible component of the tensor product of $l(\lambda)$ symmetric powers (the exponents of which can be chosen as the parts of the partition λ).

Proposition 2.2. *Let X be a compact Kähler manifold, and E, F be hermitian vector bundles on X. Suppose that E is Nakano semi-positive, and that Θ_E is positive definite at least at one point. Suppose moreover that F is Griffiths semi-positive. Then for any partition λ,*

$$H^q(X, K_X \otimes S_\lambda F \otimes (\det F)^{l(\lambda)} \otimes E) = 0 \quad \forall q > 0.$$

Proof. We first note that it is enough to treat the case of $l(\lambda) = 1$, that is, of symmetric powers. Indeed, if we can prove the vanishing for one symmetric power, then by replacing F by $F^{\oplus l}$ where $l = l(\lambda)$, which is also Griffiths semi-positive, we deduce the vanishing for a tensor product of l symmetric powers. Note that any Schur power $S_\lambda E$ is a direct summand of such a tensor product.

Now let $\pi : Y = \mathbb{P}(F^*) \to X$ be the bundle of hyperplanes of F, and $\mathcal{O}_F(1)$ be the tautological quotient line bundle on Y. By the isomorphisms [8]

$$H^q(X, K_X \otimes S^l F \otimes \det F \otimes E) \cong H^q(Y, K_Y \otimes \mathcal{O}_F(l + r) \otimes \pi^* E).$$

We want to show that $\mathcal{O}_F(m) \otimes \pi^* E$ is Nakano semi-positive, and its curvature is positive definite at least at one point, for any $m > 0$, so that we can apply Nakano vanishing theorem. Of course, we endow this vector bundle with two obvious metrics: on the one hand by the given metric on E, which we simply pull back to one on $\pi^* E$, on the other hand by the pulled-back metric on $\pi^* F$, which provides a metric on its quotient line bundle $\mathcal{O}_F(1)$. Note that, since Griffiths semi-positivity is preserved by pull-backs and quotients, this metric on $\mathcal{O}_F(1)$ has Griffiths semi-positive curvature. Moreover, the Fubini–Study metric being positive on the projective space, it has positive curvature on the fibers of π, that is, on "vertical" tangent vectors.

Now the induced metric on $\mathcal{O}_F(m) \otimes \pi^* E$ has curvature

$$\Theta_{\mathcal{O}_F(m) \otimes \pi^* E} = m \Theta_{\mathcal{O}_F(1)} \otimes Id_{\pi^* E} + \pi^* \Theta_E.$$

Let us choose a local frame $(e_1, \ldots, e_r)$ of E in a neighborhood of $x = \pi(y)$ where $y \in Y$. Then if $u = \sum_{i=1}^r u_i \otimes e_i \in T_y Y \otimes E_{\pi(y)}$, we have

$$\Theta_{\mathcal{O}_F(m) \otimes \pi^* E}(u) = m \sum_{i=1}^r \Theta_{\mathcal{O}_F(1)}(u_i) + \Theta_E(\pi_* u).$$

This is a sum of non-negative terms. Moreover, suppose that Θ_E is positive definite at $x = \pi(y)$. Then if $\Theta_{\mathcal{O}_F(m) \otimes \pi^* E}(u)$ is equal to zero, we must have $\pi_* u = 0$. This means that u has only vertical components, and since $\Theta_{\mathcal{O}_F(1)}$ is positive definite on the fibers of π, we must have $u_i = 0$ for each i, hence $u = 0$. Thus $\Theta_{\mathcal{O}_F(m) \otimes \pi^* E}$ is positive definite at y, which proves our claim. □

We will use the following vanishing theorem in the sequel, which is a slight generalization of Le Potier vanishing theorem [13]:

PROPOSITION 2.3. *Let $E_1, \ldots, E_m$ be globally generated vector bundles of ranks $e_1, \ldots, e_m$, and L be an ample line bundle, on a nonsingular complex projective variety X. Then*

$$H^q(X, K_X \otimes \wedge^{e_1-k_1} E_1 \otimes \cdots \otimes \wedge^{e_m-k_m} E_m \otimes L) = 0$$

for $q > k_1 + \cdots + k_m$.

2.3. The curvature of homogeneous bundles

We now consider the very specific case of a homogeneous vector bundle $E = G \times_P E_0$ of rank r on a generalized flag manifold $X = G/P$, where P is a parabolic subgroup of a semisimple complex Lie group G. Here E_0 is a finite dimensional P-module.

Assume that E is generated by its global sections so that the evaluation P-module homomorphism

$$\phi : V = H^0(X, E) \to E_0$$

is surjective. For a given hermitian metric on V, we have an induced metric on the fiber E_0, and a left-invariant hermitian metric on E.

The curvature of this metric was computed by Snow [18]. We need some notation to state his formula. Firstly, let us fix a maximal torus and a Borel subgroup in G, from which we deduce a root system for its Lie algebra. Up to conjugacy, the parabolic subgroup P of G is then generated by this Borel subgroup, and a closed set of certain positive roots. The opposite of these roots form a set I of simple roots, and Φ_X, called the roots of X, is then defined as the set of positive roots which, when decomposed as a linear combination of simple roots, have positive coefficients on the roots in I.

Let $v_1, \ldots, v_m$ be a basis of root vectors for V and $e_1, \ldots, e_r$ be a basis for E_0, such that

$$\phi(v_k) = \begin{cases} e_k & \text{for } 1 \leqslant k \leqslant r \\ 0 & \text{for } r < k \leqslant m. \end{cases}$$

PROPOSITION 2.4. *The curvature of the hermitian metric induced on E by the evaluation morphism ϕ, is given at the identity coset by the following formula:*

$$\Theta_E = \sum_{\alpha, \beta \in \Phi_X} \sum_{k > r} \mathrm{d}x_\alpha \wedge \mathrm{d}\bar{x}_\beta \otimes \phi(X_\alpha \cdot v_k) \otimes \overline{\phi(X_\beta \cdot v_k)}^{\mathrm{t}},$$

where transposition is taken with respect to the given basis.

Remark. Being induced by a constant metric on a trivial vector bundle, this metric on E is Griffiths semi-positive. But in general, is neither Griffiths positive, nor Nakano semi-positive and *a fortiori* not even Nakano positive. It would be useful to find conditions on E_0 ensuring that the induced metric on E has some of these properties.

3. SYMPLECTIC GRASSMANNIANS

In this section, we consider a symplectic grassmannian $\mathbb{LG}_n$, defined as the subvariety of the usual grassmannian $\mathbb{G}_{n,2n}$, consisting of maximal isotropic spaces in $\mathbb{C}^{2n}$ with respect to a fixed symplectic form. For example, $\mathbb{LG}_1 = \mathbb{P}^1$, and $\mathbb{LG}_2$ is a hyperplane section of $\mathbb{G}_{2,4}$ in its Plücker embedding, hence a three dimensional quadric. In general, $\mathbb{LG}_n$ is a hermitian symmetric space of dimension $n(n+1)/2$, homogeneous under the action of the symplectic group $Sp(2n,\mathbb{C})$. If we write $\mathbb{LG}_n = Sp(2n,\mathbb{C})/P$, then the parabolic subgroup P is defined (up to conjugacy) by the (unique) long simple root.

3.1. A Castelnuovo–Mumford criterion

We will need a Castelnuovo–Mumford criterion on the symplectic grassmannian $\mathbb{LG}_n$, that is, a cohomological criterion for a coherent sheaf on $\mathbb{LG}_n$ to be generated by its global sections. On the usual grassmannian, such a criterion is deduced in [9, 15] from the simple fact that a general section of a some homogeneous bundle (precisely, the sum of a suitable number of copies of the tautological quotient vector bundle) vanishes precisely at one point. Unfortunately, we cannot give the same argument on symplectic grassmannians (see [6] for a discussion of closely related questions). Moreover, previously mentioned Castelnuovo–Mumford criterion on $\mathbb{G}_{n,2n}$, which we may also use on the subvariety $\mathbb{LG}_n$, will not be good enough for our purposes.

Let Q_n be the tautological quotient bundle of rank n on $\mathbb{LG}_n$. We define

$$\mathscr{L}_i^{n,j} = \begin{cases} \wedge^i Q_n^* & \text{if} \quad i<j \\ \mathscr{O}_{\mathbb{LG}_n}(-1) & \text{if} \quad i=j \\ 0 & \text{if} \quad i>j. \end{cases}$$

Our Castelnuovo–Mumford criterion on $\mathbb{LG}_n$ will be the following:

Proposition 3.1. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{LG}_n$. Then $\mathscr{F}$ is generated by its global sections as soon as*

$$H^q(\mathbb{LG}_n, \mathscr{L}_{i_1}^{n,1} \otimes \cdots \otimes \mathscr{L}_{i_n}^{n,n} \otimes \mathscr{F}) = 0$$

for $q \geqslant i_1 + \cdots + i_n > 0$ and $q > i_1 + \cdots + i_n = 0$.

Remark. In the following proof and in the sequel, we will repeatedly use the following simple observation: for a coherent sheaf $\mathscr{G}$ on a variety X, let $\mathscr{R}_\bullet \to \mathscr{G} \to 0$ be a finite resolution of $\mathscr{G}$ indexed by non-negative integers. Then the cohomology group $H^q(X,\mathscr{G})$ vanishes as soon as

$$H^{q+m}(X,\mathscr{R}_m) = 0$$

for all $m \geqslant 0$. This can be easily seen by breaking the resolution into short exact sequences, and inspecting the corresponding long exact sequences of cohomology groups. Similarly, if we have a resolution $0 \to \mathscr{G} \to \mathscr{S}_\bullet$, then $H^q(X,\mathscr{G})$ vanishes as soon as $H^{q-m}(X,\mathscr{S}_m) = 0$ for all $m \geqslant 0$.

Proof. We proceed by induction on n. The space of global sections of Q_n is $\mathbb{C}^{2n}$, and the zero-locus of the section of Q_n corresponding to some non-zero vector v is the space of maximal isotropic subspaces of $\mathbb{C}^{2n}$ containing v. This locus can be identified with the lagrangian grassmannian $\mathbb{LG}_{n-1}$ of maximal isotropic spaces for the induced symplectic form on $v^\perp/v \simeq \mathbb{C}^{2n-2}$. Moreover, these loci cover $\mathbb{LG}_n$ v as varies. To prove that $\mathscr{F}$ is generated by its global sections on $\mathbb{LG}_n$, it is therefore enough to check that its restriction to any such $\mathbb{LG}_{n-1}$ is generated by its global sections, and the surjectivity of the restriction map

$$H^0(\mathbb{LG}_n, \mathscr{F}) \to H^0(\mathbb{LG}_{n-1}, \mathscr{F}|_{\mathbb{LG}_{n-1}}).$$

By the induction hypothesis, $\mathscr{F}|_{\mathbb{LG}_{n-1}}$ will be generated by its global sections as soon as

$$H^q(\mathbb{LG}_{n-1}, \mathscr{L}_{i_1}^{n-1,1} \otimes \cdots \otimes \mathscr{L}_{i_{n-1}}^{n-1,n-1} \otimes \mathscr{F}|_{\mathbb{LG}_{n-1}}) = 0$$

for $q \geqslant i_1 + \cdots + i_{n-1} > 0$ and $q > i_1 + \cdots + i_{n-1} = 0$. From the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{LG}_{n-1}} \to Q_n^*|_{\mathbb{LG}_{n-1}} \to Q_{n-1}^* \to 0$$

we get long exact sequences

$$0 \to \mathscr{O}_{\mathbb{LG}_{n-1}} \to Q_n^*|_{\mathbb{LG}_{n-1}} \to \wedge^2 Q_n^*|_{\mathbb{LG}_{n-1}} \to \cdots \to \wedge^k Q_n^*|_{\mathbb{LG}_{n-1}} \to \wedge^k Q_{n-1}^* \to 0,$$

from which we deduce that the proceeding vanishing will hold if

$$H^q(\mathbb{LG}_{n-1}, (\mathscr{L}_{j_1}^{n,1} \otimes \cdots \otimes \mathscr{L}_{j_{n-1}}^{n,n-1} \otimes \mathscr{F})|_{\mathbb{LG}_{n-1}}) = 0$$

for $q \geqslant j_1 + \cdots + j_{n-1} > 0$ and $q > j_1 + \cdots + j_{n-1} = 0$. Now we compare cohomology groups of coherent sheaves on $\mathbb{LG}_n$, and of their restriction to $\mathbb{LG}_{n-1}$, owing to the exact Koszul complex of the section of Q_n defining $\mathbb{LG}_{n-1}$:

$$0 \to \mathscr{O}_{\mathbb{LG}_n}(-1) \to \wedge^{n-1} Q_n^* \to \cdots \to Q_n^* \to \mathscr{O}_{\mathbb{LG}_n} \to \mathscr{O}_{\mathbb{LG}_{n-1}} \to 0.$$

The above vanishing on $\mathbb{LG}_{n-1}$ will thus be a consequence of the following ones on $\mathbb{LG}_n$:

$$H^q(\mathbb{LG}_n, \mathscr{L}_{j_1}^{n,1} \otimes \cdots \otimes \mathscr{L}_{j_{n-1}}^{n,n-1} \otimes \mathscr{L}_{j_n}^{n,n} \otimes \mathscr{F}) = 0$$

for $q \geqslant j_1 + \cdots + j_n > 0$ and $q > j_1 + \cdots + j_n = 0$. This is precisely our criterion. Finally, again, by the use of the above Koszul complex, the surjectivity of the restriction map

$$H^0(\mathbb{LG}_n, \mathscr{F}) \to H^0(\mathbb{LG}_{n-1}, \mathscr{F}|_{\mathbb{LG}_{n-1}})$$

is a consequence of the vanishing of $H^q(\mathbb{LG}_n, \mathscr{L}_q^{n,n} \otimes \mathscr{F})$ for $q > 0$, which is also included in our criterion. □

3.2. The key observation

If we try to apply the previous Castelnuovo–Mumford criterion on $\mathbb{LG}_n$ to $\mathscr{E}(-1)$ where $\mathscr{E}$ is the vector bundle associated to some finite covering, the classical vanishing theorem of Griffiths or Le Potier will not suffice. Actually, we will need our variant of Griffiths vanishing Theorem 2.2, which involves a Nakano positive vector bundle. Our key observation is the following:

Proposition 3.2. *The vector bundle $Q_n(1)$ on $\mathbb{LG}_n$ is Nakano positive.*

Remark. Note that the corresponding statement for the tautological quotient bundle Q on a usual grassmannian is false. However, since Q is a quotient bundle of a trivial bundle which is Griffiths semi-positive, it follows from general properties of hermitian bundles that $Q(1) = Q \otimes \det Q$ is Nakano semi-positive.

Proof. Let

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

be the $2n \times 2n$ matrix associated with a symplectic form on $\mathbb{C}^{2n}$, where I_n is the identity matrix. We follow the conventions and notations of [3]. In particular, we denote by $(e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1})$ the canonical basis of $\mathbb{C}^{2n}$. We have the roots of $\mathbb{LG}_n$,

$$\Phi_{\mathbb{LG}_n} = \{\varepsilon_i + \varepsilon_j : 1 \leqslant i \leqslant j \leqslant n\}$$

and the following corresponding root vectors:

$$X_{ij} = X_{\varepsilon_i+\varepsilon_j} = E_{i,-j} + E_{j,-i} \quad \text{for } 1 \leqslant i < j \leqslant n,$$
$$X_{ii} = X_{2\varepsilon_i} = E_{i,-i} \quad \text{for } 1 \leqslant i \leqslant n.$$

Let us apply Snow's formula 2.4 to Q_n, at the point of $\mathbb{LG}_n$ given by the maximal isotropic subspace of $\mathbb{C}^{2n}$ generated by $e_1, \dots, e_n$. The vectors $v_m = e_{-m}$, for $1 \leqslant m \leqslant n$, form a basis of the kernel of the evaluation map, and we have

$$\begin{aligned}\sum_{1 \leqslant i \leqslant j \leqslant n} \phi(X_{ij} \cdot v_m)\, dx_{ij} &= \sum_{1 \leqslant i < j \leqslant n} (\delta_{jm} e_i + \delta_{im} e_j)\, dx_{ij} + e_m\, dx_{mm} \\ &= \sum_{i=1}^{n} e_i\, dx_{im}\end{aligned}$$

by adopting the notation $dx_{ij} = dx_{ji}$. We then obtain the curvature forms

$$\Theta_{Q_n} = \sum_{1 \leqslant i,j,k \leqslant n} dx_{ik} \wedge d\bar{x}_{jk} \otimes e_i \otimes e_j^*,$$

$$\Theta_{\mathcal{O}_{\mathbb{LG}_n}(1)} = \operatorname{Trace} \Theta_{Q_n} = \sum_{1 \leqslant i,k \leqslant n} dx_{ik} \wedge d\bar{x}_{ik}.$$

The second of these formulae is a consequence of the first since $\mathcal{O}_{\mathbb{LG}_n}(1) = \det Q_n$. Then we have the hermitian form

$$\Theta_{Q_n(1)}(u) = \sum_{1 \leqslant i,j,k \leqslant n} (u^i_{jk}\, \bar{u}^j_{ik} + |u^j_{ik}|^2)$$

where $u^i_{jk} = u^i_{kj}$. We can rewrite this form as

$$\Theta_{Q_n(1)}(u) = \sum_{i<j<k} \Theta_{ijk}(u) + \sum_{i \neq j} \Theta_{ij}(u) + \sum_i \Theta_i(u)$$

where the different terms of this sum are hermitian forms involving disjoint sets of components of u. Specifically,

$$\begin{aligned}\Theta_{ijk}(u) &= 2\,(|u^i_{jk}|^2 + |u^j_{ik}|^2 + |u^k_{ij}|^2) \\ &\quad + u^i_{jk}\bar{u}^j_{ik} + \bar{u}^i_{jk}u^j_{ik} + u^i_{jk}\bar{u}^k_{ij} + \bar{u}^i_{jk}u^k_{ij} + u^j_{ik}\bar{u}^k_{ij} + \bar{u}^j_{ik}u^k_{ij} \\ &= |u^i_{jk} + u^j_{ik} + u^k_{ij}|^2 + |u^i_{jk}|^2 + |u^j_{ik}|^2 + |u^k_{ij}|^2.\end{aligned}$$

It is a positive-definite hermitian form in the three variables u^k_{ij}, u^i_{jk} and u^j_{ik}, while

$$\Theta_{ij}(u) = |u^i_{jj} + u^j_{ij}|^2 + 2\,|u^j_{ij}|^2, \quad \Theta_i(u) = 2\,|u^i_{ii}|^2$$

are positive-definite hermitian forms in u^j_{ij}, u^i_{jj} and u^i_{ii}, respectively. This clearly implies that the hermitian form $\Theta_{Q_n(1)}$ is positive definite, and the proposition is proved. □

3.3. A lemma on tensor products

Before proving Theorem A, we need the following simple remark on Schur powers, which is an easy consequence of Pieri's rules:

LEMMA 3.3. *Let $S_\lambda V$ be an irreducible component of a tensor product of m symmetric powers $S^{k_1}V \otimes \cdots \otimes S^{k_m}V$. Then the length of λ is at most m, and if it is equal to m, $S_\lambda V$ is also an irreducible component of the tensor product $\wedge^m V \otimes S^{k_1-1}V \otimes \cdots \otimes S^{k_m-1}V$.*

Remark. More precisely, $S_\lambda V$ has the same multiplicity in both tensor products, but we do not need this fact in this context.

Proof. Because of Pieri's rules, $S_\lambda V$ is a component of $S^{k_1}V\otimes\cdots\otimes S^{k_m}V$ if and only if there exists a semistandard tableau T of shape λ and weight $\kappa=(k_1,\dots,k_m)$: that is, a numbering of the diagram of the partition λ, weakly increasing on rows, strictly increasing on columns such that each integer i occurs k_i times; see [14].

The length of λ is the length of its first column. Since this column is numbered in a strictly increasing way by integers not exceeding m, its length is certainly bounded by m. If it is equal to m, this column must be numbered by $1,\dots,m$. If we remove the first column, we get a new semistandard tableau S, of shape μ and weight $(k_1-1,\dots,k_m-1)$. Since $S_\lambda V$ is an irreducible component of $\wedge^m V\otimes S_\mu V$, the lemma is proved. □

3.4. Proof of Theorem A

Let $f\colon Y\to\mathbb{LG}_n$ be a degree d covering, and $\mathscr{E}$ be the associated vector bundle on $\mathbb{LG}_n$ of rank $d-1$. We want to show that $\mathscr{E}(-1)$ is generated by its global sections. Using Serre duality and the fact that $K_{\mathbb{LG}_n}=\mathscr{O}_{\mathbb{LG}_n}(-n-1)$, we see that our Castelnuovo–Mumford criterion is equivalent to the vanishings of

$$H^{N-q}(\mathbb{LG}_n,(\mathscr{L}^{n,1}_{i_1}\otimes\cdots\otimes\mathscr{L}^{n,n}_{i_n})^*\otimes\mathscr{E}^*(-n))$$

for $q\geqslant i_1+\cdots+i_n>0$ or $q>i_1+\cdots+i_n=0$, where $N=\dim\mathbb{LG}_n$. Because of the definition of $\mathscr{E}$, Serre duality again and Leray spectral sequence, these vanishings are equivalent to the isomorphisms

$$H^q(\mathbb{LG}_n,K_{\mathbb{LG}_n}\otimes\mathscr{L}^{n,1}_{i_1}\otimes\cdots\otimes\mathscr{L}^{n,n}_{i_n}(n))\ \cong H^q(Y,K_Y\otimes f^*(\mathscr{L}^{n,1}_{i_1}\otimes\cdots\otimes\mathscr{L}^{n,n}_{i_n}(n)))$$

for $q\geqslant i_1+\cdots+i_n>0$ or $q>i_1+\cdots+i_n=0$.

If $i_k=k$ for $1\leqslant k\leqslant n$, and $q=i_1+\cdots+i_n=N$, both cohomology groups are equal to $\mathbb{C}$. If $i_k=0$ for $1\leqslant k\leqslant n$, then both groups vanish by Kodaira vanishing theorem. We shall show that all the other groups also vanish.

We first deal with cohomology groups on $\mathbb{LG}_n$. Note that they could be computed using representation theory and applying the Borel–Weil–Bott theorem, but we will give a quite different argument mainly using Griffiths' vanishing theorem and its variant in Section 2.2.

Let us consider a given n-tuple $i_1,\dots,i_n$, with $0\leqslant i_k\leqslant k$. Let a be the number of indices k such that $i_k<k$, which is supposed to be positive. Note that the sum of the other i_k's is bigger than or equal to $(n-a)(n-a+1)/2$. It will therefore be enough to prove that

$$H^q(\mathbb{LG}_n,K_{\mathbb{LG}_n}\otimes\wedge^{l_1}Q_n^*\otimes\cdots\otimes\wedge^{l_a}Q_n^*(a))=0$$

for $q\geqslant l_1+\cdots+l_a+(n-a)\ (n-a+1)/2$, and for all sequences $l_1,\dots,l_a$ with $l_k<n-a+k$. (This condition is equivalent to the existence of a strictly increasing sequence $1\leqslant m_1<\cdots<m_a\leqslant n$ such that $l_k<m_k$ for all k.)

We then use the tautological exact sequence $0\to Q_n^*\to V\to Q_n\to 0$ on $\mathbb{LG}_n$ where $V=\mathbb{C}^{2n}$ (identified with the corresponding trivial bundle on $\mathbb{LG}_n$; recall that if W is maximal isotropic subspace of V, then the quotient V/W is canonically identified with the dual W^*), and its induced long exact sequences of wedge powers

$$0\to\wedge^l Q_n^*\to\wedge^l V\to\cdots\to\wedge^{l-m}V\otimes S^mQ_n\to\cdots\to S^lQ_n\to 0.$$

First case: $a<n$. Using the above long exact sequences for each wedge power, we are reduced to prove that

$$H^{q-m_1-\cdots-m_a}(\mathbb{LG}_n,K_{\mathbb{LG}_n}\otimes S^{m_1}Q_n\otimes\cdots\otimes S^{m_a}Q_n(a))=0$$

for $q \geqslant l_1 + \cdots + l_a + (n-a)(n-a+1)/2$, and for all sequences $m_1, \dots, m_a$ with $m_k \leqslant l_k$ for all k.

Let $S_\lambda Q_n$ be an irreducible component of $S^{m_1}Q_n \otimes \cdots \otimes S^{m_a}Q_n$. If $l(\lambda) < a$, then the vanishing follows from Griffiths' vanishing theorem, more precisely from its variant 2.2 by setting $F = Q_n$, which is Griffiths semi-positive as a quotient of a trivial bundle, and by setting $E = \mathcal{O}_{\mathbb{LG}_n}(1)$, which is positive. If $l(\lambda) = a$, then Lemma 3.3 implies that it is enough to prove that

$$H^{q-m_1-\cdots-m_a}(\mathbb{LG}_n, K_{\mathbb{LG}_n} \otimes S^{m_1-1}Q_n \otimes \cdots \otimes S^{m_a-1}Q_n \otimes \wedge^a Q_n(a)) = 0.$$

Because $\wedge^a Q_n = \wedge^{n-a} Q_n^*(1)$, we can use the above complexes again to reduce to the vanishings of

$$H^{q-m_1-\cdots-m_a-m_{a+1}}(\mathbb{LG}_n, K_{\mathbb{LG}_n} \otimes S^{m_1-1}Q_n \otimes \cdots \otimes S^{m_a-1}Q_n \otimes S^{m_{a+1}}Q_n(a+1))$$

with $m_{a+1} \leqslant n-a$. We then use the same argument, repeated $n-a-1$ times, and we are finally reduced to prove the vanishings of

$$H^{q-m_1-\cdots-m_{n-1}}(\mathbb{LG}_n, K_{\mathbb{LG}_n} \otimes S^{m_1-n+a+1}Q_n \otimes \cdots \otimes S^{m_a-n+a+1}Q_n \otimes \dots$$
$$\otimes S^{m_{a+1}-n+a+2}Q_n \otimes \cdots \otimes S^{m_{n-1}}Q_n(n-1)),$$

where $m_{a+k} \leqslant n-a-k+1$. Firstly note that $q - m_1 - \cdots - m_{n-1}$ is positive since $m_1 + \cdots + m_a \leqslant l_1 + \cdots + l_a$ and $m_{a+1} + \cdots + m_{n-1} < (n-a)(n-a+1)/2$. Recall moreover that $l_1 \leqslant n-a$, so that $m_1 - n + a + 1 \leqslant 1$. If it is equal to zero, we can apply Griffiths' vanishing theorem as we did above. If it is equal to one, we can factor out a $Q_n(1)$ and apply Proposition 2.2 by setting $F = Q_n$ and $E = Q_n(1)$.

Second case: $a = n$. We need to prove that

$$H^q(\mathbb{LG}_n, K_{\mathbb{LG}_n} \otimes \wedge^{l_1}Q_n^* \otimes \cdots \otimes \wedge^{l_n}Q_n^*(n)) = 0$$

for $q \geqslant l_1 + \cdots + l_n > 0$, and for all sequences $l_1, \dots, l_n$ with $l_k < k$. Note that in particular $l_1 = 0$. Let us choose p such that $l_p > 0$. We then use our long exact sequences for each wedge power other than $\wedge^{l_p}Q_n^*$. We are thus reduced to prove that

$$H^q(\mathbb{LG}_n, K_{\mathbb{LG}_n} \otimes S^{m_2}Q_n \otimes \cdots \otimes S^{m_n}Q_n \otimes \wedge^{n-l_p}Q_n(n-1)) = 0$$

for $q \geqslant l_p$, and for all sequences $m_1, \dots, m_{p-1}, m_{p+1}, \dots, m_n$ with $m_k \leqslant l_k$. Note that we have only $n-2$ symmetric powers in this tensor product. Let $S_\lambda Q_n$ be some component of it. If $l(\lambda) < n-2$, or $l(\lambda) = n$, we can apply Griffiths' vanishing theorem. If $l(\lambda) = n-1$, than the last non-zero part of the partition λ must be equal to one. Let μ be the partition obtained by deletion of this last part: then $l(\mu) = n-2$, and $S_\lambda Q_n$ is a component of $S_\mu Q_n \otimes Q_n$. We can therefore apply Proposition 2.2 by setting $F = Q_n$ and $E = Q_n(1)$ to get the required vanishing.

There remains to prove the similar vanishings on the finite covering Y of $\mathbb{LG}_n$. But the very same argument "pulls back" to Y. Indeed, we used nothing more on $\mathbb{LG}_n$ than the tautological exact sequence, the Griffiths semi-positivity of Q_n and the Nakano positivity of $\mathcal{O}_{\mathbb{LG}_n}(1)$ and $Q_n(1)$. But the tautological sequence is clearly preserved on Y, as well as the Griffiths and Nakano semi-positivity of pulled-back bundles. Strict positivity, of course, is not preserved by pull-back. But for a finite covering, it is preserved outside the ramification locus, since outside this locus we have a local isomorphism for the complex topology. Since only positivity at one point is required to apply the vanishing theorem 2.2, we conclude our theorem. □

By proceeding as in [12], we obtain the following homotopy result with a weaker homotopy bound than the cohomology bound in Theorem B.

PROPOSITION 3.4. *Let Y and f be as in Theorem A. Then for any point $y \in Y$,*

$$f_* : \pi_i(Y, y) \to \pi_i(\mathbb{LG}_n, f(y))$$

is an isomorphism for $i \leqslant \dim \mathbb{LG}_n - \max\{d-1, k+1\}$ *where k is the k-ampleness of the tangent bundle* $T_{\mathbb{LG}_n}$.

Proof. Let $H := \mathbb{LG}_n$. Consider $f \times f$: $Y \times Y \to H \times H$, and let Δ_Y and Δ_H be the diagrams in $Y \times Y$ and $H \times H$, repectively. Using the fact that $\mathscr{E}(-1)$ is generated by its global sections on H, we see that the inclusion $\Delta_Y \hookrightarrow X := (f \times f)^{-1}(\Delta_H)$ induces a morphism

$$H_i(Y, \mathbb{Z}) \to H_i(X, \mathbb{Z})$$

which is an isomorphism for $i \leqslant \dim H - d$ and is surjective for $i = \dim H - d + 1$ (cf. [12, Lemma 5.4]). Note that Y is simply connected if $\dim H \leqslant d$ [4, Corollaire 2.6]. Now we consider the following commutative diagram of $\mathbb{Z}$-homology sequences induced by $f \times f$:

$$\begin{array}{ccccccccc}
\to & H_{i+1}(Y \times Y, X) & \to & H_i(X) & \to & H_i(Y \times Y) & \to & H_i(Y \times Y, X) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & H_{i+1}(H \times H, H) & \to & H_i(\Delta_H) & \to & H_i(H \times H) & \to & H_i(H \times H, \Delta_H) & \to
\end{array}$$

The first vertical homomorphism is an isomorphism if $i + 1 \leqslant \dim H - k$ by a version of Barth–Lefschetz theorem due to Sommese–Van de Ven [21]. Since H has homology groups in even degrees only and it doesn't have torsion, the induced morphism

$$f_* : H_i(Y, \mathbb{Z}) \to H_i(H, \mathbb{Z})$$

is an isomorphism for $i \leqslant \dim H - \max\{d-1, k+1\}$. By J.H.C. Whitehead's theorem [22], we conclude the proposition. □

4. QUADRICS

4.1. Spinor bundles

The n-dimensional nonsingular quadric $\mathbb{Q}_n$ is a homogeneous space $SO(n+2, \mathbb{C})/P$, where P is the maximal parabolic subgroup associated, in the notation of [3], with the first simple root of $so(n+2, \mathbb{C})$. It is a simple Lie algebra of type B if n is odd, and of type D if n is even. Note that the semisimple part of $Lie(P)$ is isomorphic to $so(n, \mathbb{C})$. If $n = 2m+1$, this semisimple Lie algebra has a spinor representation which defines a homogeneous bundle S on $\mathbb{Q}_n$ of rank 2^m. If $n = 2m$, there are two non isomorphic half spinor representations, giving rise to two homogeneous vector bundles S' and S'' of rank 2^{m-1}.

These *spinor bundles* have been extensively studied by Ottaviani [17]. Among other things, he showed the following:

On $\mathbb{Q}_{2m+1}$, we have a tautological exact sequence

$$0 \to S \to V \to S(1) \to 0$$

and an isomorphism $S^* \cong S(1)$ where V is trivial of rank 2^{m+1}.

On $\mathbb{Q}_{2m}$, we have two tautological exact sequences (dual to each other)

$$0 \to S' \to U \to S''(1) \to 0, \quad 0 \to S'' \to U^* \to S'(1) \to 0,$$

where U is trivial of rank 2^m. If $m \equiv 0 \pmod 2$ then $S'^* \simeq S'(1)$ and $S''^* \simeq S''(1)$, and if $m \equiv 1 \pmod 2$ then $S'^* \cong S''(1)$ and $S''^* \cong S'(1)$.

Furthermore, spinor bundles behave nicely under restriction to generic hyperplane sections: namely, the restriction of S to $\mathbb{Q}_{2m}$ is $S' \oplus S''$, while the restriction of S' or S'' to $\mathbb{Q}_{2m-1}$ is the spinor bundle on that quadric.

4.2. Quadrics of dimension five

On $\mathbb{Q}_5$, we have the following Castelnuovo–Mumford criterion:

PROPOSITION 4.1. *A coherent sheaf $\mathscr{F}$ on $\mathbb{Q}_5$ is generated by its global sections as soon as*

$$\begin{array}{lll} H^{i+j}(\mathbb{Q}_5, \mathscr{F}(-l)) = 0 & \textit{for } (i,l) = (1,1), (2,2), (5,3), & j \geqslant 0 \\ H^{i+j}(\mathbb{Q}_5, \mathscr{F} \otimes S(-l)) = 0 & \textit{for } (i,l) = (1,0), & j \geqslant 0 \\ H^{i}(\mathbb{Q}_5, \mathscr{F} \otimes \wedge^2 S(-l)) = 0 & \textit{for } (i,l) = (3,1), (4,2). & \end{array}$$

Proof. Let x be a point in $\mathbb{Q}_5$, and $\mathbb{Q}_4$ a four-dimensional quadric in $\mathbb{Q}_5$ containing x. Since $\mathbb{Q}_4$ is a hyperplane section, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{Q}_5}(-1) \to \mathcal{O}_{\mathbb{Q}_5} \to \mathcal{O}_{\mathbb{Q}_4} \to 0$$

showing in particular that the restriction map

$$H^0(\mathbb{Q}_5, \mathscr{F}) \to H^0(\mathbb{Q}_4, \mathscr{F}|_{\mathbb{Q}_4})$$

is surjective, since $H^1(\mathbb{Q}_5, \mathscr{F}(-1)) = 0$ by hypothesis.

On $\mathbb{Q}_4$, we have two half spinor bundles S' and S'' of rank 2, and x can be realized as the zero-locus of a section of $(S' \oplus S')^*$. Indeed, we can identify $\mathbb{Q}_4$ with the grassmannian $\mathbb{G}_{2,4}$ in such a way that S'^* identifies with the tautological quotient bundle on that grassmannian. Using the Koszul complex associated with a section of $(S' \oplus S')^*$, we see that $\mathscr{F}|_{\mathbb{Q}_4}$ is globally generated at x as soon as

$$H^q(\mathbb{Q}_4, \wedge^q(S' \oplus S') \otimes \mathscr{F}|_{\mathbb{Q}_4}) = 0 \quad \forall q > 0.$$

Now recall that $S|_{\mathbb{Q}_4} = S' \oplus S''$, so that for $q = 1$, this vanishing is a consequence of the following conditions on $\mathbb{Q}_5$:

$$H^1(\mathbb{Q}_5, \mathscr{F} \otimes S) \cong H^2(\mathbb{Q}_5, \mathscr{F} \otimes S(-1)) = 0$$

where the isomorphism is obtained by the tautological exact sequence on $\mathbb{Q}_5$ and our hypothesis. For $q = 2$, we want to show that $H^2(\mathbb{Q}_4, S' \otimes S' \otimes \mathscr{F}|_{\mathbb{Q}_4}) = 0$. Tensoring the second tautological sequence by $S'(-1) \otimes \mathscr{F}|_{\mathbb{Q}_4}$ on $\mathbb{Q}_4$, we reduce to the vanishings of $H^2(\mathbb{Q}_4, S'(-1) \otimes \mathscr{F}|_{\mathbb{Q}_4})$ and $H^3(\mathbb{Q}_4, S' \otimes S''(-1) \otimes \mathscr{F}|_{\mathbb{Q}_4})$. The first of these groups is dealt with as above. For the second one, we note that $S' \otimes S''$ is a summand of $\wedge^2 S|_{\mathbb{Q}_4}$, so that our vanishing condition is a consequence of

$$H^3(\mathbb{Q}_5, \mathscr{F} \otimes \wedge^2 S(-1)) = H^4(\mathbb{Q}_5, \mathscr{F} \otimes \wedge^2 S(-2)) = 0.$$

Finally, the cases $q = 3$ and $q = 4$ are similar to that of $q = 1$. □

Proof of Theorem C for $3 \leqslant n \leqslant 5$. Recall that $\mathbb{Q}_3 \simeq \mathbb{LG}_2$ and $\mathbb{Q}_4 \simeq \mathbb{G}_{2,4}$, so that Theorem C follows for $n = 3$ from Theorem A, and for $n = 4$ from [15]. Let now $n = 5$.

Let $f\colon Y \to Q_5$ be a branched covering and $\mathscr{E}$ be the associated vector bundle. We prove that $\mathscr{E}(-1)$ is generated by global sections, using our Castelnuovo–Mumford criterion on $\mathbb{Q}_5$. Using Serre duality, the definition of $\mathscr{E}$ and the fact that $K_{\mathbb{Q}_5} = \mathscr{O}_{\mathbb{Q}_5}(-5)$, we see that what we need to show is that certain cohomology groups of type

$$H^i(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes \wedge^j S(4-l))$$

are equal to the corresponding cohomology groups of the pulled-back bundles on Y. We show that all these groups vanish, with one exception.

This is clear for $j = 0$ since l is at most two, which allows to use Kodaira vanishing. For $j = 1$, we can use the tautological exact sequence $0 \to S \to V \to S^* = S(1) \to 0$, which easily implies that

$$H^i(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes S(m)) \cong H^{i-n}(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes S(m+n))$$

provided that m, $m+n$ and $i-n$ are positive. Since $S = S^*(-1)$, $\det S^* = \mathscr{O}_{\mathbb{Q}_5}(2)$ and S^* is globally generated, the Griffiths vanishing theorem then gives

$$H^i(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes S(4-l)) = H^1(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes S^* \otimes \det S^*(i-l)) = 0$$

for $0 \leqslant l < i \leqslant 4$, which is the case in our criterion.

Finally, for $j = 2$, we have two conditions. The first one concerns

$$H^3(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes \wedge^2 S(3)) \cong H^3(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes \wedge^2 S^*(1))$$

which is zero by the Le Potier vanishing theorem. For the second one, we shall prove that

$$H^4(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes \wedge^2 S(2)) \cong C.$$

We first notice that S^* is globally generated, but $c_4(S^*) = 0$; see [17]. This implies that a generic section of S^* does not vanish anywhere. Choosing such a section, we get an exact sequence

$$0 \to \mathscr{O}_{\mathbb{Q}_5} \to S^* \to Q^* \to 0$$

where Q^* is a globally generated vector bundle and $\det Q^* = \mathscr{O}_{\mathbb{Q}_5}(2)$. Moreover, there is an exact sequence $0 \to \wedge^2 Q \to \wedge^2 S \to Q \to 0$. For $i \geqslant 3$, we have $H^i(K_{\mathbb{Q}_5} \otimes Q(2)) \cong H^i(K_{\mathbb{Q}_5} \otimes S(2))$, which we proved to be zero. Hence

$$H^4(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes \wedge^2 S(2)) \cong H^4(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes \wedge^2 Q(2)) \cong H^4(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes Q^*).$$

And since $H^i(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes S^*) = H^i(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes S(1)) = 0$ for $i \geqslant 4$, we have

$$H^4(\mathbb{Q}_5, K_{\mathbb{Q}_5} \otimes Q^*) \cong H^5(\mathbb{Q}_5, K_{\mathbb{Q}_5}) = \mathbb{C}$$

as claimed. Since the argument "pulls back" to Y by f, we are done. □

4.3. Quadrics of dimension six

On $\mathbb{Q}_6$ the situation is very close to that of $\mathbb{Q}_5$, except that we have now two half spinor bundles S' and S'', both of rank four, both having for restriction to $\mathbb{Q}_5$ the spinor bundle S on this subquadric. Since $\mathbb{Q}_5$ is a hyperplane section of $\mathbb{Q}_6$, the exact sequence $0 \to \mathscr{O}_{\mathbb{Q}_6}(-1) \to \mathscr{O}_{\mathbb{Q}_6} \to \mathscr{O}_{\mathbb{Q}_5} \to 0$ allows to deduce almost immediately from 4.1, a Castelnuovo-Mumford criterion on $\mathbb{Q}_6$: a coherent sheaf $\mathscr{F}$ on $\mathbb{Q}_6$ will be generated by its global sections as soon as

$$\begin{array}{lll} H^{i+j}(\mathbb{Q}_6, \mathscr{F}(-l)) = 0 & \textit{for} \quad (i,l) = (1,1), (2,2), (4,3), (6,4) & j \geqslant 0 \\ H^{i+j}(\mathbb{Q}_6, \mathscr{F} \otimes S'(-l)) = 0 & \textit{for} \quad (i,l) = (1,0), (2,1), & j \geqslant 0 \\ H^{i}(\mathbb{Q}_6, \mathscr{F} \otimes \wedge^2 S'(-l)) = 0 & \textit{for} \quad (i,l) = (3,1), (4,2), (5,3). & \end{array}$$

Proof of Theorem C for $n = 6$. The argument is almost exactly the same as on $\mathbb{Q}_5$. Indeed, we have $c_4(S') = c_4(S'') = 0$ [17]. Choosing a nowhere vanishing section of S'^*, we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{Q}_6} \to S'^* \to Q'^* \to 0$$

where Q'^* is a globally generated vector bundle and $\det Q'^* = \mathcal{O}_{\mathbb{Q}_6}(2)$. Using this sequence, we check that

$$H^5(\mathbb{Q}_6, K_{\mathbb{Q}_6} \otimes \wedge^2 S'(2)) = H^5(\mathbb{Q}_6, K_{\mathbb{Q}_6} \otimes Q'^*) = H^5(\mathbb{Q}_6, K_{\mathbb{Q}_6}) = \mathbb{C}.$$

This is enough to get the result that if $f\colon Y \to \mathbb{Q}_6$ is a branched covering with its associated vector bundle $\mathscr{E}$, then $\mathscr{E}(-1)$ is generated by its global sections, since the other cohomology groups involved in our Castelnuovo-Mumford criterion on $\mathbb{Q}_6$ are easily checked to vanish. $\square$

5. GLOBAL GENERATION

In this section we prove that for any branched covering of ordinary and symplectic flag manifolds, the associated vector bundle $\mathscr{E}$ is generated by its global sections. Of course, $\mathscr{E}$ cannot be ample in general, since flag manifolds that are not grassmannians fiber over smaller flag manifolds, from which one could pull-back any branched covering. Nevertheless, they should have intermediate positivity properties, presumably some k-ampleness in the sense of Sommese. This would again imply Barth–Lefschetz-type theorems, but unfortunately we have been unable to prove such positivity properties.

5.1. Ordinary flag manifolds

Let $\mathbb{F}_n$ denote the variety of complete flags in $V = \mathbb{C}^n$, with universal flag

$$W_\bullet\colon\quad 0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset V \otimes \mathcal{O}_{\mathbb{F}_n}$$

of subbundles W_i of rank i of the trivial vector bundle $V \otimes \mathcal{O}_{\mathbb{F}_n}$. We denote by $Q_i = V/W_{n-i}$ the quotient bundle of rank i, and

$$\mathcal{O}_{\mathbb{F}_n}(a_1, \ldots, a_{n-1}) = \mathcal{O}_{\mathbb{F}_n}\left(\sum_{i=1}^{n-1} a_i \varepsilon_i\right) = \bigotimes_{i=1}^{n-1} (\det Q_i)^{a_i}.$$

This line bundle is generated by its global sections (ample, respectively) if and only if $a_i \geqslant 0$ ($a_i > 0$, respectively) for all i. The canonical line bundle of $\mathbb{F}_n$ is $K_{\mathbb{F}_n} = \mathcal{O}_{\mathbb{F}_n}(-2, \ldots, -2)$.

Let x be a point of $\mathbb{F}_n$, given by some complete flag $0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset V$. Let us choose a compatible basis $u_1, \ldots, u_n$ of V, that is such that $u_1, \ldots, u_i$ is a basis of U_i for all i. Each vector u_{n-i} defines a global section of the quotient vector bundle Q_i. We therefore have a global section of $Q_1 \oplus \cdots \oplus Q_{n-1}$, vanishing precisely at x. The corresponding Koszul complex is

$$0 \to \wedge^{(n(n-1))/2}(Q_1^* \oplus \cdots \oplus Q_{n-1}^*) \to \cdots \to Q_1^* \oplus \cdots \oplus Q_{n-1}^* \to \mathcal{O}_{\mathbb{F}_n} \to \mathcal{O}_x \to 0.$$

Hence, we obtain a very simple Castelnuovo–Mumford criterion on $\mathbb{F}_n$:

PROPOSITION 5.1. *A coherent sheaf $\mathscr{F}$ on $\mathbb{F}_n$ is generated by its global sections as soon as*

$$H^q(\mathbb{F}_n, \wedge^{i_1} Q_1^* \otimes \cdots \otimes \wedge^{i_{n-1}} Q_{n-1}^* \otimes \mathscr{F}) = 0$$

for $q = i_1 + \cdots + i_{n-1} > 0$.

We will apply this criterion to the vector bundle associated with a finite covering of $\mathbb{F}_n$. With the generalized Le Potier Vanishing Theorem 2.3 in mind, it will be easy to prove the first half of Theorem D:

PROPOSITION 5.2. *Let $f: Y \to \mathbb{F}_n$ be a finite surjective morphism where Y is a nonsingular connected complex projective variety. Then the associated vector bundle $\mathscr{E}$ is generated by its global sections.*

Proof. We need to prove that

$$H^q(\mathbb{F}_n, K_{\mathbb{F}_n} \otimes \wedge^{i_1} Q_1^* \otimes \cdots \otimes \wedge^{i_{n-1}} Q_{n-1}^*(2, \ldots, 2) \otimes \mathscr{E}) = 0$$

for $q = i_1 + \cdots + i_{n-1} > 0$. Using Serre duality, the definition of $\mathscr{E}$, and Leray spectral sequence, this vanishing is seen to be equivalent to the equality between the cohomology group

$$H^q(\mathbb{F}_n, K_{\mathbb{F}_n} \otimes \wedge^{i_1} Q_1^* \otimes \cdots \otimes \wedge^{i_{n-1}} Q_{n-1}^*(2, \ldots, 2)),$$

and the corresponding cohomology group on Y,

$$H^q(Y, K_Y \otimes f^*(\wedge^{i_1} Q_1^* \otimes \cdots \otimes \wedge^{i_{n-1}} Q_{n-1}^*(2, \ldots, 2)))$$

for $q = i_1 + \cdots + i_{n-1} > 0$. We shall prove that both groups are equal to zero. We begin with the first one.

If $i_1 > 0$, then since Q_1 is of rank 1 we can suppose that $i_1 = 1$, and we apply the generalized Le Potier Vanishing Theorem 2.3. If $i_1 = 0$, let k be the smallest integer such that $i_k > 0$. We proceed by induction on k. We have a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{F}_n}(\varepsilon_k - \varepsilon_{k-1}) \to Q_k \to Q_{k-1} \to 0$$

from which we get an exact sequence for wedge powers:

$$0 \to \wedge^{i_k} Q_{k-1}^* \to \wedge^{i_k} Q_k^* \to \wedge^{i_k - 1} Q_{k-1}^*(\varepsilon_{k-1} - \varepsilon_k) \to 0.$$

Let us tensor this exact sequence by $K_{\mathbb{F}_n} \otimes \wedge^{i_k - 1} Q_1^* \otimes \cdots \otimes \wedge^{i_{n-1}} Q_{n-1}^*(2, \ldots, 2)$. The qth cohomology group of the first vector bundle of the exact sequence we get, then vanishes by induction hypothesis. Also, for the last vector bundle we obtain the cohomology group

$$H^q(\mathbb{F}_n, K_{\mathbb{F}_n} \otimes \wedge^{i_k - 1} Q_{k-1}^* \otimes \wedge^{i_k - 1} Q_{k+1}^* \cdots \otimes \wedge^{i_{n-1}} Q_{n-1}^*(2, \ldots, 3, 1, 2, \ldots,, 2))$$

where 3 and 1 occupy the $(k-1)$th and kth position, respectively. But this is zero by the generalized Le Potier vanishing theorem. The qth cohomology group of the middle vector bundle therefore vanishes, which is what we wanted to prove.

Since the very same argument applies to the pulled-back vector bundles on Y, the proposition is proved. □

COROLLARY 5.3. *Let $f: Y \to F$ be a finite covering of any variety of incomplete flags F, where Y is a nonsingular connected complex projective variety. Then the associated vector bundle $\mathscr{E}$ is generated by its global sections.*

Proof. Since a variety of complete flags fibers on any variety of incomplete flags of the same vector space, we have a smooth fibration $\pi: \mathbb{F}_n \to \mathbb{F}$. Pulling back by f, we get a finite covering $f': Y' \to \mathbb{F}_n$ with its associated vector bundle $\mathscr{E}' = \pi^* \mathscr{E}$. Moreover, since π has connected fibers, $\pi_* \mathcal{O}_{\mathbb{F}_n} = \mathcal{O}_{\mathbb{F}_n}$, hence

$$H^0(Y', \mathscr{E}') = H^0(Y, \pi_* \mathscr{E}') = H^0(Y, \mathscr{E}).$$

Since, by the proposition above, $\mathcal{E}'$ is generated by its global sections, all of which are pull-backs of sections of $\mathcal{E}$, then $\mathcal{E}$ itself must be globally generated. □

5.2. *Symplectic flag manifolds*

We denote by $\mathbb{LF}_n$ the variety of complete lagrangian flags in $V = \mathbb{C}^{2n}$, with universal flag of subbundles

$$W_\bullet: \quad 0 = W_0 \subset W_1 \subset \cdots \subset W_n = W_n^\perp \subset W_{n-1}^\perp \subset \cdots \subset W_1^\perp \subset \mathcal{O}_{\mathbb{LG}_n} \otimes V,$$

where $\dim W_i = i$ and $\dim W_i^\perp = 2n - i$. This variety has dimension n^2, and is homogeneous under the natural action of the symplectic group $Sp(2n, \mathbb{C})$. We will consider it as a subvariety of $\mathbb{F}_{n,2n}$, the variety of incomplete flags in $\mathbb{C}^{2n}$ consisting of subspaces of dimensions 1 to n.

For $1 \leqslant i \leqslant n$, we denote by $Q_i = V/W_i$ the quotient bundle of rank $2n - i$, and

$$\mathcal{O}_{\mathbb{LF}_n}(a_1, \ldots, a_n) = \mathcal{O}_{\mathbb{F}_n}\left(\sum_i a_i \varepsilon_i\right) = \bigotimes_{i=1}^{n} (\det Q_i)^{a_i}.$$

In particular, the canonical line bundle of $\mathbb{LF}_n$ is $K_{\mathbb{LF}_n} = \mathcal{O}_{\mathbb{LF}_n}(-2, \ldots, -2)$.

On $\mathbb{F}_{n,2n}$, a generic section of $Q_1 \oplus \cdots \oplus Q_n$ vanishes at a simple point, and the associated Koszul complex provides us with a Castelnuovo–Mumford criterion, which can be restricted to $\mathbb{LF}_n$. Since we have $Q_i^* \simeq W_{n-i+1}^\perp$ on $\mathbb{LF}_n$, we get:

PROPOSITION 5.4. *A coherent sheaf $\mathcal{F}$ on $\mathbb{LF}_n$ is generated by its global sections as soon as*

$$H^q(\mathbb{LF}_n, \wedge^{i_1} W_1^\perp \otimes \cdots \otimes \wedge^{i_n} W_n^\perp \otimes \mathcal{F}) = 0$$

for $q = i_1 + \cdots + i_n > 0$.

In the sequel, we will need to understand the cohomology of certain homogeneous bundles, which will be tensor products of wedge powers of the tautological vector bundles. We call such a bundle a *wedge bundle*, and say that it is of *type* $t = (t_1, \ldots, t_n)$ if it has t_i factors that are wedge powers of W_i or $W_i^\perp$, and *weight* $w = (w_1, \ldots, w_n)$ if the sum of the exponents of these wedge powers of W_i and $W_i^\perp$ is equal to w_i. The type can of course be decomposed into subtypes t' and t'', given by the number of factors of involving W_i, and $W_i^\perp$, respectively. We first need the following refined statement of the generalized Le Potier vanishing theorems for wedge bundles on $\mathbb{LF}_n$.

PROPOSITION 5.5. *Let $\mathcal{W}$ be a wedge bundle on $\mathbb{LF}_n$, of type t and weight w. Then*

$$H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \mathcal{W}(l_1, \ldots, l_n)) = 0$$

for $q > w_1 + \cdots + w_n$, if either:

- $l_k > t_k$ *for* $1 \leqslant k \leqslant n$,
- $l_k > t_k$ *for* $k \neq p, q$, *where* $p < q$ *are such that* $t''_q > 0$, $l_q \geqslant t_q$ *and* $l_p > t_p + 1$, *or*
- $l_k > t_k$ *for* $k \neq p, q$, *where* $p > q$ *are such that* $t'_q > 0$, $l_q \geqslant t_q$ *and* $l_p > t_p + 1$.

Proof. The first assertion is an immediate consequence of the generalized Le Potier vanishing theorem. Indeed, we can rewrite each wedge power $\wedge^m W_i$ as $\wedge^{n-i+1-m} W_i^*(-1)$, and similarly each $\wedge^m W_i^\perp$ as $\wedge^{n-i+1-m}(W_i^\perp)^*(-1)$. If $l_k > t_k$ for $1 \leqslant k \leqslant n$, we get a tensor product of wedge powers of globally generated vector bundles, by the ample line bundle $\mathcal{O}_{\mathbb{LF}_n}(l_1 - t_1, \ldots, l_n - t_n)$. Hence, we can apply the generalized Le Potier Vanishing Theorem 2.3, and we obtain the desired statement.

For the second assertion, which is not a direct consequence of Le Potier's theorem, we will proceed by induction on q. Since t_q'' is non-zero, then we have at least a factor $\wedge^m W_q^\perp$ in $\mathscr{W}$. Therefore, we can use the exact sequence

$$0 \to \wedge^m W_q^\perp \to \wedge^m W_{q-1}^\perp \to \wedge^{m-1} W_q^\perp(\varepsilon_q - \varepsilon_{q-1}) \to 0.$$

Let us tensor this exact sequence by the other factors of $\mathscr{W}$. We get a short exact sequence involving wedge bundles,

$$0 \to \mathscr{W} \to \mathscr{W}' \to \mathscr{W}''(\varepsilon_q - \varepsilon_{q-1}) \to 0.$$

Here $\mathscr{W}'$ has weight $(w_1, \ldots, w_{q-1} + m, w_q - m, \ldots, w_n)$ and type $(t_1, \ldots, t_{q-1} + 1, t_q - 1, \ldots, w_n)$. Hence

$$H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \mathscr{W}'(l_1, \ldots, l_n)) = 0$$

for $q > w_1 + \cdots + w_n$. Indeed, this is the case by induction hypothesis if $p < q - 1$, while if $p = q - 1$ we are in the situation of the first part of the proposition. On the other hand, $\mathscr{W}''$ has weight $(w_1, \ldots, w_{q-1}, w_q - 1, \ldots, w_n)$ and type $(t_1, \ldots, t_{q-1}, t_q, \ldots, w_n)$ (except if $m = 1$, in which case t_q is replaced by $t_q - 1$, which is even better). This again allows us to use our induction hypothesis, which gives

$$H^{q-1}(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \mathscr{W}''(l_1, \ldots, l_{q-1} - 1, l_q + 1, \ldots, l_n)) = 0$$

for $q > w_1 + \cdots + w_n$. And these vanishings for $\mathscr{W}'$ and $\mathscr{W}''$ imply our claim for $\mathscr{W}$.

The third assertion is proved in the very same way using the other series of exact sequences,

$$0 \to \wedge^m W_q \to \wedge^m W_{q+1} \to \wedge^{m-1} W_q(\varepsilon_q - \varepsilon_{q+1}) \to 0$$

and a similar, but now descending, induction on q. □

With the help of this vanishing theorem, we can now extend Vanishing Theorem 5.3 to the case of symplectic flag manifolds and prove the second half of Theorem D. As for ordinary flags, it will suffice to consider complete symplectic flags, that is, coverings of $\mathbb{LF}_n$.

Proposition 5.6. *Let $f\colon Y \to \mathbb{LF}_n$ be a finite covering, where Y is a nonsingular connected complex projective variety. Then the associated vector bundle $\mathscr{E}$ is generated by its global sections.*

Proof. We apply our Castelnuovo–Mumford criterion on $\mathbb{LF}_n$ to $\mathscr{E}$. Because of Serre duality and the definition of $\mathscr{E}$, we need to prove that the cohomology group

$$H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \wedge^{i_1} W_1^\perp \otimes \cdots \otimes \wedge^{i_n} W_n^\perp(2, \ldots, 2))$$

for $q = i_1 + \cdots + i_n > 0$, is equal to the corresponding group obtained by pull-back to Y. We shall prove that both groups are equal to zero. Note that on $\mathbb{LF}_n$, this would follow from the first part of the proceeding proposition if we had W_1 instead of $W_1^\perp$, at least for $i_1 > 0$. Indeed, since W_1 has rank one, this would force $i_1 = 1$ and we could replace W_1 by $\mathcal{O}_{\mathbb{LF}_n}(-\varepsilon_1)$. Our strategy will therefore be to use the tautological exact sequences on $\mathbb{LF}_n$, first to replace $W_1^\perp$ by $W_n^\perp = W_n$, then W_n by W_1.

Let k be the smallest integer such that $i_k > 0$. For notational simplicity, let us suppose that $k = 1$: the argument would be the same for $k > 1$. We have an exact sequence

$$0 \to \wedge^{i_1} W_2^{\perp} \to \wedge^{i_1} W_1^{\perp} \to \wedge^{i_1 - 1} W_2^{\perp}(\varepsilon_2 - \varepsilon_1) \to 0.$$

Let us tensor it by $\mathscr{W} = \wedge^{i_2} W_2^{\perp} \otimes \cdots \otimes \wedge^{i_n} W_n^{\perp}$. The wedge bundle $\mathscr{W} \otimes \wedge^{i_1 - 1} W_2^{\perp} (\varepsilon_2 - \varepsilon_1)$ has type $(0, 2, 1, \ldots, 1)$; so that by the second part of Proposition 5.5,

$$H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \mathscr{W} \otimes \wedge^{i_1 - 1} W_2^{\perp}(1, 3, 2, \ldots, 2) = 0.$$

Our vanishing will therefore follow from that of $H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \wedge^{i_1} W_2^{\perp} \otimes \mathscr{W}(2, \ldots, 2))$. Using the same argument $n - 1$ times, we are then reduced to prove that

$$H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \wedge^{i_1} W_n^{\perp} \otimes \mathscr{W}(2, \ldots, 2)) = 0.$$

But since $W_n^{\perp} = W_n$, we can now use the exact sequence

$$0 \to \wedge^{i_1} W_{n-1} \to \wedge^{i_1} W_n \to \wedge^{i_1 - 1} W_{n-1}(\varepsilon_{n-1} - \varepsilon_n) \to 0.$$

Using the third part of Proposition 5.5, we see that

$$H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \wedge^{i_1} W_{n-1} \otimes \mathscr{W}(2, \ldots, 2, 3, 1)) = 0.$$

We are thus reduced to prove that $H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \wedge^{i_1} W_{n-1} \otimes \mathscr{W}(2, \ldots, 2)) = 0$ and, using repeatedly the same argument, what we finally have to show is the vanishing of

$$H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \wedge^{i_1} W_1 \otimes \mathscr{W}(2, \ldots, 2)).$$

This is clear if $i_1 > 1$. If $i_1 = 1$, we can rewrite this cohomology group as

$$H^q(\mathbb{LF}_n, K_{\mathbb{LF}_n} \otimes \mathscr{W}(1, 2, \ldots, 2)),$$

which is zero by the first part of Proposition 5.5. □

5.3. Questions and conjectures

Let $X = G/P$ be a complex projective homogeneous space, where G is a complex semisimple Lie group and P is a parabolic subgroup. If P is not maximal and Q is a parabolic subgroup of G containing P, we have a smooth fibration $G/P \to G/Q$. Then we obtain a branched covering f of X by pulling back the one of G/Q. Note that its associated bundle, being a pull-back by f, cannot be ample: it is at best $(\dim Q/P)$ -ample. Let us define

$$k_X = \max_{Q \supset P} \dim Q/P.$$

If P is defined (up to conjugacy) by a set I of simple roots, one can compute k_X as the maximum, as α describes I, of the number of positive roots having positive coefficient on the simple root α, but zero coefficient on the other simple roots in I.

We can then extend the conjecture of Debarre stated in the introduction in the following naive way:

Conjecture *Let $f: Y \to X = G/P$ be a branched covering, with Y smooth and connected. Then the associated vector bundle $\mathscr{E}$ is k_X-ample.*

Once again, this would imply a Barth–Lefschetz-type theorem: the natural map

$$f^*: H^i(X, \mathbb{C}) \to H^i(Y, \mathbb{C})$$

would then be an isomorphism for $i \leqslant \dim X - k_X - d + 1$, where d is the degree of f. Moreover, one could expect this k_X-ampleness to be optimal only when f is obtained by pull-back:

QUESTION. *Let $f: Y \to X = G/P$ be a branched covering, with a nonsingular connected complex projective variety Y. Suppose that the associated vector bundle $\mathscr{E}$ is not $(k_X - 1)$-ample. Then, is f necessarily a pull-back of a covering of some smaller homogeneous space G/Q?*

To attack the previous conjecture by the methods of this paper, we would need *efficient* Castelnuovo–Mumford criteria. Even when P is maximal, we could not always find good enough such criteria, for example on quadrics, or on spinor varieties, which being hermitian symmetric should be easier to deal with than orthogonal grassmannians.

QUESTION. *How to find "good" Castelnuovo–Mumford criteria on homogeneous spaces?*

When the parabolic group P is not maximal, such criteria should certainly involve several line bundles. A special case would be the following:

QUESTION. *Let L, M be globally generated line bundles on a projective variety X, with $L \otimes M$ very ample. If $\mathscr{F}$ is a coherent sheaf on X, can one give a cohomological criterion for the surjectivity of the mixed evaluation morphism*

$$H^0(X, \mathscr{F} \otimes L^{-1}) \otimes L \oplus H^0(X, \mathscr{F} \otimes M^{-1}) \otimes M \to \mathscr{F}$$

that would not necessarily imply that $\mathscr{F} \otimes L^{-1}$ or $\mathscr{F} \otimes M^{-1}$ is globally generated?

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